International Journal of Research in Advent Technology, Vol.7, No.1, January 2019
E-ISSN: 2321-9637
Available online at www.ijrat.org

Near and Closer Relations in Bitopology

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Abstract- Dvalishvili studied the concepts of near relations and closer relations in topology in 2005. In 2009, these relations were further investigated by Thamizharasi and Thangavelu. Recently the authors introduced the weak forms of near and closer relations in topology. In this paper, the weak forms of near and closer relations in bitopology have been discussed .

Keywords: Near relations, closer relations, topology, bitopology, b-open sets, semi-open sets.
2010 AMS Subject Classification: 54A05, 54A10, 54E55.

1. INTRODUCTION AND PRELIMINARIES

Topologists extended the notions of semi-open, α-open, pre-open, β-open ,b-open and $b^{\#}$-open sets in topology to bitopological spaces. The concepts of near relations and closer relations in topology that were discussed in [5, 12]. These notions are further investigated in [10]. Recently the authors introduced the weak forms of near and closer relations in topology. The purpose of this paper is to investigate these relations in bitopological settings. Throughout this paper (X,τ) is a topological space and (X, τ_1, τ_2) is a bitopological space, i,j=1,2 and $i\neq j$. Also A and B are the subsets of X. $Cl_i A$ = the closure of A and $Int_i A$ = the interior of A with respect to τ_i.

Definition 2.1: A is called
(i).regular open [9] if $A= Int\, Cl A$
(ii).semiopen[6] if there exists an open set U with $U\subseteq A\subseteq Cl U$
(iii). preopen [7] if there exists an open set U with $A\subseteq U\subseteq Cl A$
(iv).b -open [2] if $A\subseteq Cl\, Int A\cup Int\, Cl A$
(vi).$b^{\#}$ -open [11] if $A\subseteq Cl\, Int A\cup Int\, Cl A$
(vii).α-open [7] if $A\subseteq Int\, Cl\, Int A$.
(viii).β-open [1] if $A\subseteq Cl\, Int\, Cl\, A$.

The Complements of sets in Definition 2.1 are called the corresponding closed sets. The corresponding interior and closure operators can be defined in the usual manner. The following lemma will be useful in sequel.

Lemma 2.2:
(i).sIntA = $A\cap Cl\, Int A$ and sClA = $A\cup Int\, Cl\, A$.
(ii).pIntA = $A\cap Int\, Cl\, A$ and pClA = $A\cup Cl\, Int\, A$.
(iii).αIntA = $A\cap Cl\, Int\, Cl\, A$ and αClA = $A\cup Cl\, Int\, Cl\, A$.
(iv).βIntA = $A\cap Cl\, Int\, Cl\, A$ and βClA = $A\cup Int\, Cl\, Int\, A$.

Definition 2.3: A is called
(i).ij-regular open [10] if $A= Int_i Cl_j A$
(ii).ij-semiopen[10] if there exists an i-open set U with $U\subseteq A\subseteq Cl_j U$
(iii).ij-preopen [10] if there exists an i-open set U with $A\subseteq U\subseteq Cl_j A$
(iv).ij-b -open [10] if $A\subseteq Cl_j Int_i A\cup Int_i Cl_j A$
(v).ij-$b^{\#}$ -open [3] if $A=Cl_j Int_i A\cup Int_i Cl_j A$
(vi).ij-α-open [10] if $A\subseteq Int_i Cl_j Int_i A$.
(vii).ij-β -open [10] if $A\subseteq Cl_i Int_j Cl_i A$.

The Complements of sets in Definition 2.3 are called the corresponding closed sets. Further A is ij-regular closed if and only if $A = Cl_j Int_i A$. The following lemmas have been established in [10].

Lemma 2.5: A is
(i).ij-semiopen$\Leftrightarrow A \subseteq Cl_j Int_i A$.
(ii).ij-preopen $\Leftrightarrow A \subseteq Int_i Cl_j A$.
(iii).ij-b-closed $\Leftrightarrow Int_j Cl_i A \cap Cl_i Int_j A \subseteq A$,
The concepts of $sInt_{ij}A$, and $sCl_{ij}A$ can be defined in a usual way.

Lemma 2.6:
(i) $sInt_{ij} A = A \cap Cl_j Int_i A$
(ii) $sCl_{ij} A = A \cup Int_j Cl_i A$

3. NEAR RELATIONS IN BITOPOLOGY

The next proposition shows that Lemma 2.2 can be established in bitological settings.

Proposition 3.1:
(i).A is ij-α-open iff $A = A \cap Int_i Cl_j Int_i A$ and A is ij-α-closed iff $A = A \cup Cl_i Int_j Cl_i A$.
(ii).A is ij-preopen iff $A = A \cap Int_i Cl_j A$ and A is ij-preclosed iff $A = A \cup Cl_i Int_j A$.
(iii).A is ij-β-open iff $A = A \cap Cl_i Int_j Cl_i A$ and A is ij-β-closed iff $A = A \cup Int_i Cl_j Int_i A$.
.

Proof: Suppose A is ij-α-open. Then $A \subseteq Int_i Cl_j Int_i A$ that implies $A = A \cap Int_i Cl_j Int_i A$. Conversely let $A = A \cap Int_i Cl_j Int_i A$. Then $A \subseteq Int_i Cl_j Int_i A$ that implies that A is ij-α-open. This proves the first part of (i). Now suppose A is ij-α-closed. Then $A \supseteq Cl_i Int_j Cl_i A$ that implies $A = A \cup Cl_i Int_j Cl_i A$. Conversely let $A = A \cup Cl_i Int_j Cl_i A$. Then $A \supseteq Cl_i Int_j Cl_i A$ that implies A is ij-α-closed. This proves (i). Other results in the proposition can be analogously established.

Remark 3.2: The above proposition motivates to define the closure and interior operators of a bitoplogical space in the following manner.

Definition 3.3:
(i).$pInt_{ij} A = A \cap Int_i Cl_j A$ and $pCl_{ij} A = A \cup Cl_i Int_j A$.
(ii).$\alpha Int_{ij} A = A \cap Int_i Cl_j Int_i A$ and $\alpha Cl_{ij} A = A \cup Cl_i Int_j Cl_i A$.
(iii).$\beta Int_{ij} A = A \cap Cl_i Int_j Cl_i A$ and $\beta Cl_{ij} A = A \cup Int_i Cl_j Int_i A$.

Definition 3.4: We say that
(i).A is ij-near to B if $Int_i A = Int_j B$
(ii).A is ij-seminear to B if $sInt_i A = sInt_j B$
(iii).A is ij-α-near to B if $\alpha Int_i A = \alpha Int_j B$
(iv). A is ij-prenear to B if $pInt_i A = pInt_j B$
(v).A is ij-β-near to B if $\beta Int_i A = \beta Int_j B$

Proposition 3.5:
(i).A is ij-near to B if and only if B is ji-near to A.
(ii).A is ij-seminear to B if and only if B is ji-seminear to A.
(iii).A is ij-α-near to B if and only if B is ji-α-near to A.
(iv).A ij- is prenear to B if and only if B is ji-prenear to A.
(v).A is ij-β-near to B if and only if B is ji-β-near to A.

Proof: Straight forward.

Proposition 3.6: Let A and B be any two subsets of X such that $Int_i A \in \tau_j$ and $Int_j B \in \tau_i$. If A is ij-seminear or ij-α-near or ij-pre-near or ij-β-near to B then A is ij-near to B.

Proof: Suppose A is ij-seminear to B. Then $sInt_i A = sInt_j B$ that implies
$A \cap Cl_i Int_i A = sInt_i A = sInt_j B = B \cap Cl_j Int_j B$. Now $Int_i A \subseteq sInt_i A = sInt_j B = B \cap Cl_j Int_j B \subseteq B$
that implies $Int_i A \subseteq B$. Since $Int_i A \in \tau_j$ it follows that $Int_i A \subseteq Int_j B$. Similarly we can prove that $Int_j B \subseteq Int_i A$. This proves that $Int_i A = Int_j B$ that implies A is ij-near to B. The other cases can be analogously proved.

Proposition 3.7:
(i).If A is ij-regular open then A is i-near to $Cl_j A$.
(ii).If A is ij-semiclosed or ij-α-closed then A is j-near to $Cl_i A$.
(iii).If A is ij-preclosed then A is j-near to $Cl_i Int_j A$.
(iv).If A is β-closed or ij-b-closed or ij-$b^{\#}$-closed then A is i-near to $Cl_j Int_i A$.

Proof: Suppose A is ij-regular open. Then A = Int_iCl_jA that implies $Int_iA = Int_iCl_jA$. This proves that A is near to Cl_jA in (X,τ_i). This proves (i). If A is ij-semiclosed then $Int_jA = Int_jCl_iA$ that implies A is j-near to Cl_iA. If A is ij-α-closed then $Cl_iInt_jCl_iA\subseteq A$ that implies $Int_jA=Int_jCl_i\ Int_jCl_iA = Int_jCl_iA$ that proves that A is j-near to Cl_iA. This proves (ii). Suppose A is ij-preclosed. Then $Cl_iInt_jA\subseteq$ A that implies $Int_jA = Int_jCl_i\ Int_j$ A that proves that A is j-near to Cl_iInt_jA. This proves (iii) . Suppose A is ij-β-closed. Then $Int_iCl_jInt_iA\subseteq A$ that implies $Int_iA = Int_iCl_jInt_i$ A that proves that A is i-near to Cl_jInt_iA. Suppose A is ij-b-closed or ij-b$^\#$-closed then $A\supseteq Int_iCl_jA\cap Cl_jInt_iA$ that implies $Int_iA\supseteq Int_iCl_jA\cap Int_iCl_jInt_iA = Int_i\ Cl_jInt_iA \supseteq Int_iA$ that proves that $Int_iA=Int_i\ Cl_j\ Int_i$ A. This proves that A is i-near to $Cl_j\ Int_i$ A. This proves (iv).

Lemma 3.8: Let A be ij-near to B and C be ij-near to D. Then

(i) $A\cap C$ is ij-near to $B\cap D$.

(ii) $A\cup C$ is not ij-near to $B\cup D$.

Proof: Since A is ij-near to B and since C is ij-near to D, $Int_iA = Int_j$ B and $Int_iC = Int_j$ D. This implies $Int_iA\cap Int_iC = Int_j\ B\cap Int_jD$ that implies $Int_i(A\cap C) = Int_j(B\cap D)$. This proves that $A\cap C$ is ij-near to $B\cap D$ that implies (i). Since $Int_i(A\cup C) \neq Int_iA\cup Int_iC$, it follows that $A\cup C$ is not ij-near to $B\cup D$.

It is easy to see that the relation " is ij-near to " is not reflexive. Further this relation is neither symmetric nor transitive.

Lemma 3.9: Let O_i be i-open and O_j be j-open in (X,τ_1,τ_2) .

(i) O_i is ij-near to B if and only if $O_i=Int_jB$.

(ii) A is ij-near to O_j if and only if $Int_iA = O_j$.

Proposition 3.10:

(i). For each i-open set O_i, B={$B\subseteq X$: O_i is ij-near to B} is a base for some topology on Y where Y is the union of all members of B.

(ii). For each j-open set O_j, A={$A\subseteq X$: A is ij-near to O_j} is a base for some topology on Z where Z is the union of all members of A.

Proof: Let Y be the union of members of B . Suppose $A\in B$ and $B\in B$. Then O_i is ij-near to $A\cap B$. This implies B is a base for some topology $\tau(O_i)$ on Y. Similarly we can prove that A is a base for some topology $\tau(O_j)$ on Z.

Remark 3.11: The topologies $\tau(O_i)$ and $\tau(O_j)$ can be extended to the topologies on X. In fact $\tau(O_i)\cup\{X\}$and $\tau(O_j)\cup\{X\}$ are topologies on X induced by the i-open set O_i and j-open set O_j respectively.

Remark 3.12: Every pair (O_i, O_j) of open sets induces a bitopology on X. In particular (O_1, O_2) and (O_2, O_1) induce bitopologies on X.

Proposition 3.13: Let A be ij-near to B and $B\subseteq A$. Then

(i).If A is ij-semiopen then B is j-semiopen.

(ii).If A is ij-α-open then B is j-semiopen.

(iii).If B is ij-preclosed then A is i-preclosed.

Proof: Suppose A is ij-semiopen. Then $A\subseteq Cl_jInt_iA$. Since A is ij-near to B, $Int_i A= Int_jB$ that implies $B\subseteq A\subseteq Cl_jInt_iA=Cl_jInt_jB$. This proves that B is j-semiopen that proves (i).

Suppose A is ij-α-open. Then $A\subseteq Int_iCl_jInt_i$ A. Since A is ij-near to B, $Int_iA = Int_jB$ that implies $B\subseteq A \subseteq Int_iCl_jInt_iA = Int_i\ Cl_jInt_jB\subseteq Cl_jInt_jB$. This proves that B is j-semiopen that proves (ii). Suppose B is ij-preclosed. Then $B\supseteq Cl_iInt_jB$. Since A is ij-near to B, $Int_iA = Int_jB$ that implies $A\supseteq B\supseteq Cl_i\ Int_jB =Cl_iInt_iA$. This proves that A is i-preclosed that proves (iii).

4. CLOSER RELATIONS IN BITOPOLOGY

Definition 4.1: We say that A is ij-closer (resp. ij-semicloser, resp. ij-α-closer, resp. ij-precloser, resp. ij-β-closer) to B if $Cl_iA=Cl_jB$ (resp. $sCl_iA=sCl_jB$, resp. $\alpha Cl_iA=\alpha Cl_jB$, resp. $pCl_iA=pCl_jB$, resp. $\beta Cl_iA=\beta Cl_jB$).

Proposition 4.2:

(i).A is ij-closer to B if and only if B is ji-closer to A.

(ii).A is ij-semi-closer to B if and only if B is ji-semi-closer to A.

(iii).A is ij-α-closer to B if and only if B is ji-α-closer to A.

(iv).A is ij-pre-closer to B if and only if B is ji-α-closer to A.
(v).A is ij-β-closer to B if and only if B is ji-α-closer to A.
Proof: Straight forward.

Proposition 4.3: Let A and B be any two subsets of X such that $X\backslash Cl_i A \in \tau_j$ and $X\backslash Cl_j B \in \tau_i$. If A is ij-semicloser or ij-α-closer or ij-pre-closer or ij-β-closer to B then A is ij-closer to B.

Proof: Suppose A is ij-semi closer to B. Then $sCl_i A = sCl_j B$ that implies $A \cup Int_i Cl_i A = sCl_i A = sCl_j B = B \cup Int_j Cl_j B$. Now $Cl_i A \supseteq sCl_i A = sCl_j B = B \cup Int_j Cl_j B \supseteq B$ that implies $Cl_i A \supseteq B$. Since $X\backslash Cl_i A \in \tau_j$, $Cl_i A \supseteq Cl_j B$. Similarly we can prove that $Cl_j B \supseteq Cl_i A$. This proves that $Cl_i A = Cl_j B$ that implies A is ij-closer to B. The other cases can be analogously established.

Proposition 4.4:
(i).If A is ij-regular closed then A is closer to $Int_j A$ in (X,τ_i)
(ii). If A is ij-semiopen or ij-α-open then A is j-closer to $Int_i A$.
(iii).If A is ij-preopen or ij-b-open or $b^{\#}$-open then A is j-closer to $Int_i Cl_j A$.
(iv).If A is ij-β-open then A is i-closer to $Int_j A$.

Proof: Suppose A is ij-regular closed. Then $A = Cl_i Int_j A$ that implies $Cl_i A = Cl_i Int_j A$. This proves that A is i-closer to $Int_j A$. This proves (i). Suppose A is ij-semiopen. Then $Cl_j A = Cl_j Int_i A$ that implies A is j-closer $Int_i A$. Suppose A is ij-α-open. Then $A \subseteq Int_i Cl_j Int_i A$ that implies $Cl_j A = Cl_j Int_i Cl_j Int_i A$ that proves that A is j-closer to $Int_i A$. This proves (ii). Suppose A is ij-preopen. Then $A \subseteq Int_i Cl_j A$ that implies $Cl_j A = Cl_j Int_i Cl_j A$ that proves that A is j-closer to $Int_i Cl_j A$. Suppose A is ij-b-open or ij-$b^{\#}$-open. Then $A \subseteq Int_i Cl_j A \cup Cl_j Int_i A$ that implies $Cl_j A \subseteq Cl_j Int_i Cl_j A \cup Cl_j Int_i A = Cl_j Int_i Cl_j A \subseteq Cl_j A$ that proves that $Cl_j A = Cl_j Int_i Cl_j A$. This proves that A is j-closer to $Int_i Cl_j A$. This proves (iii). Suppose A is ij-β-open. Then $A \subseteq Cl_i Int_j Cl_i A$ that implies $Cl_i A = Cl_i Int_j Cl_i Int_j A = Cl_i Int_j A$ that proves that A is i-closer to $Int_j A$. This proves (iv).

Lemma 4.5: A is ij-near to B iff $X\backslash A$ is ij-closer to $X\backslash B$.
Proof: Since A is ij-near to B, $Int_i A = Int_j B$ that implies $Cl_i(X\backslash A) = Cl_j(X\backslash B)$.This proves that $X\backslash A$ is ij-closer to $X\backslash B$.The converse part is analogous.

Lemma 4.6: Let A be ij-closer to B and C be ij-closer to D. Then
(i).$A \cup C$ is ij-closer to $B \cup D$.
(ii).$A \cap C$ is not ij-closer to $B \cap D$.
Proof: Since A is ij-closer to B and C is ij-closer to D, $Cl_i A = Cl_j B$ and $Cl_i C = Cl_j D$. This implies $Cl_i A \cup Cl_i C = Cl_j B \cup Cl_j D$ that implies $Cl_i(A \cup C) = Cl_j(B \cup D)$.This proves that $A \cup C$ is ij-closer to $B \cup D$ that implies (i). Since $Cl_i (A \cap C) \neq Cl_i A \cap Cl_i C$, it follows that $A \cap C$ is not ij-closer to $B \cap D$.
This proves (ii).

Lemma 4.7: Let F_i be i-closed and F_j be j-closed in (X,τ_1,τ_2). Then
(i) F_i is ij-closer to B if and only if $F_i = Cl_j B$.
(ii) A is ij-closer to F_j if and only if $Cl_i A = F_j$
.
Proposition 4.8: Let A be ij-closer to B and $B \subseteq A$. Then
(i).If A is ji-preopen then B is j-preopen.
(ii).If B is ji-semiclosed then A is i-semiclosed.
(iii).If B is ji-α-closed then A is i-semiclosed.
Proof: Suppose A is ji-preopen. Then $A \subseteq Int_j Cl_i A$. Since A is ij-closer to B, $Cl_i A = Cl_j B$ that implies $B \subseteq A \subseteq Int_j Cl_i A = Int_j Cl_j B$. This proves that B is j-preopen that proves (i).
Suppose B is ji-semiclosed. Then $B \supseteq Int_i Cl_j B$. Since A is ij-closer to B, $Cl_i A = Cl_j B$ that implies $A \supseteq B \supseteq Int_i Cl_j B = Int_i Cl_i A$. This proves that A is i-semiclosed that proves (ii).
Suppose B is ji-α-closed. Then $B \supseteq Cl_j Int_i Cl_j B$. Since A is ij-closer to B $Cl_i A = Cl_j B$ that implies $A \supseteq B \supseteq Cl_j Int_i Cl_j B \supseteq Int_i Cl_j B = Int_i Cl_i A$. This proves that A is i-semiclosed that proves (iii).

5. CONCLUSION

Nearly open sets and nearly closed sets in bi topological spaces are characterized by using the near and closer relations introduced between the two topologies on a bitoplogical space. It has been established that every near class is a base for some topology.

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